Final exam for Quantum Physics 1 - 2013-2014

Thursday 28 November 2013, 18:30-21:30

READ THIS FIRST:

- Je mag zelf weten of je de antwoorden in het **Nederlands** of **Engels** opschrijft.
- Clearly write your name and study number on each answer sheet that you use.
- Start each question (number 1, 2, etc.) on a new answer sheet.
- On the first answer sheet, write the total number of answer sheets that you turn in.
- When turning in your answers, please stack your answer sheets in the proper order, and **staple** them together.
- The exam has several questions, it continues on the backsides of the papers.
- Note that the lower half of this page lists some useful formulas and constants.
- The exam is open book with limits. You are allowed to use the book by Griffiths, the handouts *Extra note on two-level systems and exchange degeneracy for identical particles* and *Feynman Lectures chapter III-1*, and one A4 sheet with your own notes, but nothing more than this.
- If it says "make a rough estimate", there is no need to make a detailed calculation, and making a simple estimate is good enough. If it says "calculate" or "derive", you are supposed to present a full analytical calculation.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first.
- The full exam is 80 points (4 problems of 20 points).

Useful formulas and constants:

Electron mass	$m_e = 9.1 \cdot 10^{-31}$ kg
Electron charge	$-e = -1.6 \cdot 10^{-19}$ C
Planck's constant	$h = 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs
Planck's reduced constant	$\hbar = 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs
Energy units	1 eV $= 1.6 \cdot 10^{-19}$ J

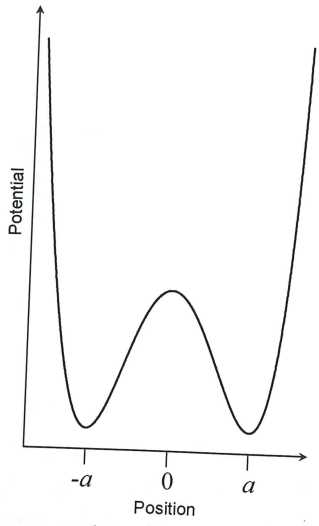
Fourier relations between x-representation and k-representation of a state

$$\overline{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, \Psi(x)\, dx$$

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, \overline{\Psi}(k)\, dk$$

Problem 1

Consider a one-dimensional quantum particle in a double-well potential (see figure).



We will assume that only the low-energy dynamics of this system is relevant. In that case, it can be described as an effective two-state system. The system then has two position eigenstates, which belong to the operator (observable) $\hat{A}$ for the position of the particle in this double-well system,

$$\hat{A} \leftrightarrow \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}, \quad |\varphi_L\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\varphi_R\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One of these states, denoted as $|\varphi_L\rangle$, corresponds to the particle being localized at $-a$ (in the left well), and has the position eigenvalue $-a$. The other position eigenstate, denoted as $|\varphi_R\rangle$ with eigenvalue $+a$ corresponds to the particle being localized in the right well. We also introduced a matrix and vector notation for representing the operators and states of this system, using $|\varphi_L\rangle$ and $|\varphi_R\rangle$ as basis vectors.

When the particle is fully localized in the left well it has a potential energy E_0. When the particle is fully localized in the right well it also feels a potential energy E_0. The particle can go from the left well to the right well by tunneling through the barrier. Using the same matrix notation as before (using $|\varphi_L\rangle$ and $|\varphi_R\rangle$ as basis vectors), the Hamiltonian for the particle is (here T is a real and negative number, and E_0 is a real positive number)

$$\hat{H} \leftrightarrow \begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}.$$

In this description, the system has two energy eigenstates, $|\varphi_g\rangle$ and $|\varphi_e\rangle$,

$$|\varphi_g\rangle \leftrightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad |\varphi_e\rangle \leftrightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

a) [1 point]

Calculate the eigenvalues that belong to $|\varphi_g\rangle$ and $|\varphi_e\rangle$.

b) [3 points]
Describe in words what the following four quantities represent, and calculate their values.

$\langle \varphi_L | \hat{H} | \varphi_L \rangle$, $\langle \varphi_R | \hat{H} | \varphi_R \rangle$, $\langle \varphi_L | \hat{H} | \varphi_R \rangle$ and $\langle \varphi_R | \hat{H} | \varphi_L \rangle$.

c) [3 points]
Describe in words what the following four quantities represent, and calculate their values.

$\langle \varphi_g | \hat{H} | \varphi_g \rangle$, $\langle \varphi_e | \hat{H} | \varphi_e \rangle$, $\langle \varphi_g | \hat{H} | \varphi_e \rangle$ and $\langle \varphi_e | \hat{H} | \varphi_g \rangle$.

d) [6 points]
Assume that the system is in the state $|\Psi_0\rangle = 3|\varphi_g\rangle + e^{i\gamma}|\varphi_e\rangle$. Note that the state is not normalized. Calculate for this case the expectation value for the position of the particle (in the sense of observable $\hat{A}$).

e) [7 points]
At time $t = 0$, the system is prepared in the state $|\varphi_L\rangle$, after which the system is free to evolve. Calculate how the following expectation values depend on time for $t > 0$.

e1) The expectation value for the total energy of the system.

e2) The expectation value for the potential energy that is present in the system.

Hint: first write down the matrix that represents the potential energy of the system.

Problem 2
Consider a quantum particle that can only move in one dimension (along the x-axis). It is a free particle.

a) [2 points]
Write down the eigenvalue equation for the position operator for this system, and shortly mention what each part or symbol in the equation represents.

b) [5 points]
Describe in words what the position eigenstates and position eigenvalues are according to the eigenvalue equation for position (if you find it useful, you can also use equations).
Also explain whether in reality the system can be prepared in a state that is such a position eigenstate. Explain how you can do it, or why it is impossible.
Answer guideline: Your full answer should be about 5-15 lines of text.

Further introduction to questions 2c)-2d)
If you would measure whether the particle is between position $x = a$ and position $x = b$, the answer can be yes or no. A commonly used expression for the probability for the measurement outcome "yes" is

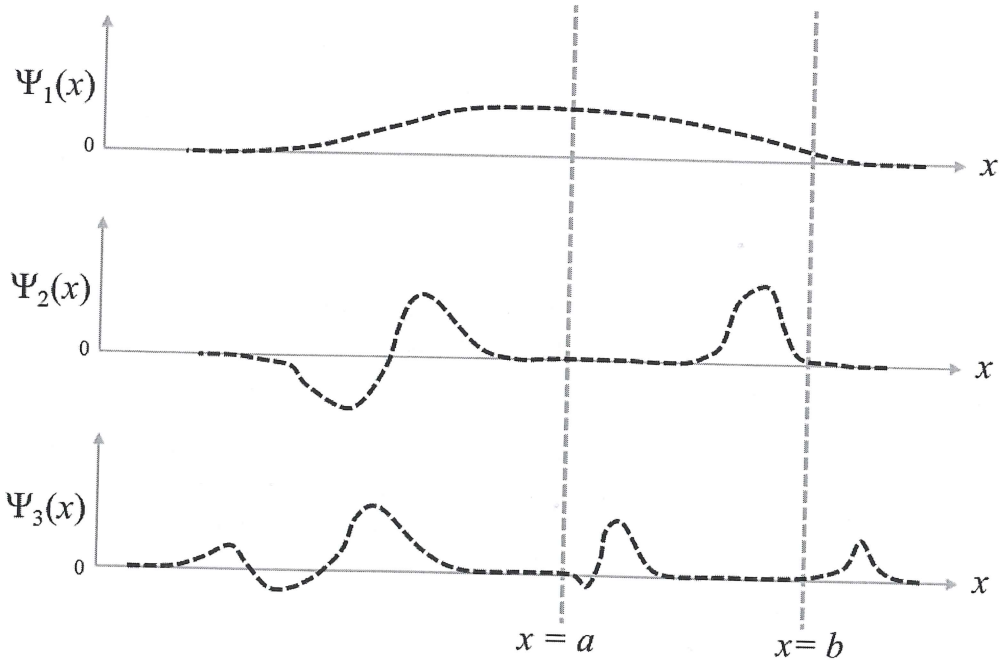
$$P_{ab} = \int_a^b \Psi(x)^* \, \Psi(x) \, dx, \qquad (1)$$

where $\Psi(x)$ is the state of the particle before the measurement.
However, the most general expression for probabilities of measurement results is

$$P_{ab} = \left| \langle \varphi_{ab} | \Psi \rangle \right|^2, \qquad (2)$$

where $|\Psi\rangle$ is the state before the measurement, and $|\varphi_{ab}\rangle$ is the state after the measurement if the measurement showed that the particle was indeed between position $x = a$ and position $x = b$.
In the remainder of this question you are asked to prove that the two expressions are in fact the same.

c) [5 points]
For proving that the two expressions are the same, it is important to first define in more details what the state $|\varphi_{ab}\rangle$ is. A good way to think about measuring whether the particle is between position $x = a$ and position $x = b$ is to consider that you pass a screen with detection pixels that has a slit in it over the x-axis with the particle (during this motion the screen is parallel to the x-axis). The slit in the screen runs from $x = a$ to $x = b$. If the screen hits the particle, the particle sticks to one of the detection pixels of the screen and its position is then visible on the screen (and you learn that the measurement outcome is not that the particle was between $x = a$ and $x = b$). But there is some probability that the screen does not hit the particle because there is a slit in the screen. We consider here the case that the slit is quite wide, such that there is effectively no interaction between the screen and the particle for almost all x-positions within the slit. In the sketch below here, you see three examples of $\Psi(x)$ (they all three happen to be real valued). Re-draw these states on your answer sheet (use dashed lines), and for each case also draw the state $|\varphi_{ab}\rangle$ (used solid lines) for the case that the screen did not hit the particle.



d) [8 points]
Now prove that expression (1) and (2) are in fact the same.
Hint: For your analysis, make a sketch of some arbitrary function $\Psi(x)$ that has non-zero amplitudes both inside and outside the interval from $x = a$ to $x = b$. Go with expression (2) from Dirac representation to x-representation, and do not forget to properly renormalize $|\varphi_{ab}\rangle$ when using it in the x-representation.

Problem 3

Consider the spin of a single electron in a magnetic field in z-direction. The magnitude of the field is B_z. The Hamiltonian for this spin is then

$$\hat{H} = -\gamma B_z \hat{S}_z ,$$

where is $\gamma = -1.761 \cdot 10^{11}$ rad s^{-1} T^{-1}, and $\hat{S}_z$ is the operator for the z-component of the spin. The operators for the x- and y-component of the spin are $\hat{S}_x$ and $\hat{S}_y$, respectively. For notation use that the states $|\uparrow\rangle$ and $|\downarrow\rangle$ represent spin-up and spin-down along the z-axis.

a) [3 point]

In a representation that uses the states $|\uparrow\rangle$ and $|\downarrow\rangle$ as basis states, write down the matrices for $\hat{S}_z$, $\hat{S}_x$, $\hat{S}_y$, and $\hat{H}$.

b) [5 point]

Calculate the eigenvalues and eigenvectors of $\hat{S}_x$ (do the calculation, and also give your answer, while using the states $|\uparrow\rangle$ and $|\downarrow\rangle$ as basis states).

c) [5 points]

The spin is prepared in the state $|\Psi_0\rangle = \sqrt{\tfrac{1}{2}}|\uparrow\rangle - i\sqrt{\tfrac{1}{2}}|\downarrow\rangle$. *Calculate* for this case the exact value for the quantum uncertainty ΔS_z in the z-component of the spin.

d) [2 points]

Calculate (or simply list the answer if you know it) all matrix elements $\langle\uparrow|\hat{S}_j|\uparrow\rangle$, $\langle\uparrow|\hat{S}_j|\downarrow\rangle$, $\langle\downarrow|\hat{S}_j|\uparrow\rangle$ and $\langle\downarrow|\hat{S}_j|\downarrow\rangle$, for the two cases $j = x,z$.

e) [5 points]

The spin is at time $t = 0$ prepared in the state $|\Psi_0\rangle = \sqrt{\tfrac{1}{2}}|\uparrow\rangle - i\sqrt{\tfrac{1}{2}}|\downarrow\rangle$. Calculate how the expectation value $\langle\hat{S}_x\rangle$ depends on time for $t > 0$.

Hint: for short notation you could use the results $\langle\uparrow|\hat{S}_j|\uparrow\rangle$ etc. of question d).

Problem 4

Consider a molecule that consists of two rotating parts that are coupled. The rotating parts of course have angular momentum. The two corresponding angular momentum vectors will be denoted as vectors $\boldsymbol{L}$ and $\boldsymbol{J}$. The angular momentum $\boldsymbol{L}$ has a magnitude $|\boldsymbol{L}| = \sqrt{12}\,\hbar$. The total angular momentum of this system (addition of all contributions) will be denoted as $\boldsymbol{F}$. There are no other contributions to the total angular momentum besides $\boldsymbol{L}$ and $\boldsymbol{J}$. The z-component of $\boldsymbol{F}$ will be denoted as F_z, with likewise notation for the z-component of $\boldsymbol{L}$ and $\boldsymbol{J}$.

a) [5 points]
Write down the 4 equations (eigenvalue problems) that determine the possible measurement outcomes for $|\boldsymbol{L}|$, L_z, $|\boldsymbol{F}|$, and F_z. If you need to introduce quantum numbers the suggested notation is l, m_l, f, and m_f, respectively. Use Dirac notation.

b) [5 points]
Consider a beam of such molecules, in which the molecules are in any of the possible angular momentum states. An instrument isolates a single molecule from the beam and measures F_z. After that, the instrument takes a next molecule out of the beam and measures again F_z. This is repeated many times. The measurement results include (in units of $\hbar$) -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6. Use these results to determine the quantum number j that is associated with the value of $|\boldsymbol{J}|$. Explain your answer.

c) [5 points]
The instrument of question b) is used on one particular molecule from the beam, and the measurement outcome is $F_z = 6\hbar$.
c1) Describe the state of the system after the measurement.
c2) Make an *estimate* (or calculate it, if you prefer that) of the expectation value for the x-component of $\boldsymbol{F}$ after the measurement.
c3) Make an *estimate* of the quantum uncertainties in the x-component and y-component of $\boldsymbol{F}$ after the measurement.
Explain your answers.

d) [5 points]
The instrument of question b) is changed to one that measures $|\boldsymbol{F}|$ instead. What measurement outcomes do you now expect when the instrument measures on a molecule from the beam? Explain your answer.